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Application of Vortex Invariants to Roll Up of Vortex Pairs

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A method developed by Betz for the rolled-up structure of vortices shed by isolated wing tips is extended to vortex pairs (two vortical regions of opposite sign), which are shed by wings of finite span. The present analysis again uses the invariants for the two-dimensional time-dependent motion of vortex systems, but the extension made here to vortex pairs depends primarily on the invariant for kinetic energy. It is found that the total energy in the flowfield can be separated into a part that governs the structure of each vortical region and a part that governs the spanwise distance between the centroids of the vortical regions. As a consequence, the rules for the rolled-up structure of vortical regions are the same as the one derived by Betz. Because the analysis does not apply a constraint that forces the circular contours of constant circulation to coincide with streamline paths, the solution is labeled first order. A method that can be used to obtain second- and higher-order approximations is described. Application of the first-order method to the vortex wake of an elliptically loaded wing indicates that a first-order solution for vortex pairs is adequate for many engineering purposes. The method derived here for single vortex pairs also justifies the superposition of axially symmetric vortical cores to simulate complex vortex wakes.

Nomenclature

b = wing span, ft (m)
b' = distance between vortex centers, ft (m)
C_L = lift coefficient, $=L/qS$
c = wing chord, ft (m)
J = second moment of circulation, ft^4/s (m^4/s)
L = lift, lb (N)
M = angular moment of circulation, ft^4/s^2 (m^4/s^2)
N = total number of vortices
q = dynamic pressure, lb/ft^2 (N/m^2)
r = radius, ft (m)
S = wing planform area, ft^2 (m^2)
T = kinetic energy in flowfield, lb-ft^4-s (N-m^4-s)
t = time, s
U_∞ = freestream velocity, ft/s (m/s)
v = spanwise velocity, ft/s (m/s)
W = Kirchhoff–Routh path function, ft-lb (N-m)
w = vertical velocity, ft/s (m/s)
x = distance in flight direction, ft (m)
y = distance in spanwise direction, ft (m)
z = distance in vertical direction, ft (m)
Γ = circulation, ft^2/s (m^2/s)
γ = vorticity, 1/s
ρ = air density, lb-s^2/ft^4 (kg/m^3)
φ = velocity potential, ft^2/s (m^2/s)
Ψ = stream function, ft^2/s (m^2/s)

Subscripts

c = inboard end of vortex segment
fs = freestream
i, j = indices
ir = irrotational part of flowfield
n = vortex number
o = centerline value
pr = plane of rolled-up vortex pair
ptvr = point vortex
ro = rotational part of flowfield
v = axially symmetric vortical region
w = wing plane
∞ = freestream quantity

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Introduction

A DIRECT relationship between the spanwise lift distribution on a semi-infinite wing, and the rolled-up structure of the vortex wake that trails behind it, was identified by Betz.[1] The method is easy to apply and quite reliable even when other vortices are in the wake.[1–3] Betz greatly simplified his analysis with the assumption that the vortex is remote enough from the other side of the wake shed by a wing of finite span, so that the streamlines, and the distribution of vorticity, in the rolled-up vortex are along concentric circles. It is also assumed that the spanwise bound vortex in the wing does not influence the roll-up process. In brief, Betz' method determines the radius of a circle in the rolled-up plane that contains the circulation shed by the segment of the vortex sheet shed by a wing between the wing tip and a specified spanwise station. The distribution of vorticity within that circle is determined by a stepwise process from the wing tip inboard to the centerline of the wing to yield the circulation in the rolled-up vortex as a function of radius, which can then be converted to vorticity. Because Betz' method provides good results in cases where it could be checked,[3–5] multiple vortices within an aircraft wake are often treated by the superposition of two or more axially symmetric vortex structures, even though the superposition process had not been validated.[6,7] Such a method is in common use because it makes it possible to predict easily the structure of complex vortex wakes when the spanwise lift distribution is known. The method has also been used extensively to interpret measurements made in lift-generated wakes.[2–5,7]

The purpose of the study being reported is to extend Betz' method to vortex wakes composed of vortex pairs. As with Betz' method, the analysis does not include the effects of the vortex sheet shed by the horizontal tail, nor of the bound vortices in the wing surface, on the structure of rolled-up vortical regions and their spacings. As before, the analysis is based on the two-dimensional time-dependent invariants for vortex systems embedded in an incompressible and inviscid fluid.[8,9] Even with these simplifications, it was not possible to develop a simple direct method that is exact for the formation of vortex pairs. It was, however, possible to obtain a first-order approximation that is simple to apply and accurate enough for many engineering purposes. The solution is called first order because it approximates the distributed circulation in the flow field with concentric circles of vorticity, that is, circles that contain a certain amount of circulation. Near the centers of the vortices, the streamline pattern and the contours of vorticity are then closely aligned. However, in parts of the flowfield not close to a vortex center the streamlines often follow the vorticity contours only approximately. The assumption that the circulation, or vorticity, elements are distributed along concentric circles is necessary in the analysis of vortex pairs in order to be able to formulate fairly simple relationships for the kinetic energy of the fluid in the flowfield.

Refinement of the vorticity/circulation distributions to better align them with the streamline pattern can be accomplished by use of second- and higher-order approximations to iteratively relocate the vortical regions, which drive the streamline paths into noncircular shapes. The revised streamline paths will usually be such that the second- and higher-order approximations must be carried out numerically. Fortunately, it is found that a first-order solution for the roll up of the vortex sheet shed by an elliptically loading wing is probably accurate enough for many engineering purposes.

The following text first describes the flowfield models to be used. The vortex invariants (which are usually written for point-vortex systems) are then written in integral form for vortical regions that contain continuous distributions of circulation/vorticity. To illustrate how the analysis of the flowfield of vortex pairs is to be carried out, an overview is presented of the solution found for isolated wing tips, first by use of the invariant for the second moment of circulation and then by use of the invariant for kinetic energy in the flowfield. The derivation of the roll-up equations for the first-order solution for the flowfield of vortex pairs is then described, along with applications, and procedures to be used for higher-order iterations to yield more exact solutions.

Wake Model for Isolated Wing Tip

The wake model used by Betz[1] for the prediction of the rolled-up structure of a vortex sheet shed by an isolated wing tip into an axially symmetric vortex is illustrated in Fig. 1a. The strength of the vortex sheet shed at the trailing edge of a wing is then directly related to variations in the span loading over the wing-tip region of a wing of semi-infinite span. Because the wing has a semi-infinite span, the rolled-up vortex is remote enough from other vortical flowfields so that the streamline paths and vorticity distributions both lay along concentric circles.

The equation presented in Figs. 1a and 1b designates the radius at which elements from the vortex sheet are placed within the rolled-up vortex. This concise equation is derived by use of the conservation of circulation and the first and second moments of circulation.[2] The simplicity of Betz' method comes about because the isolation of the rolled-up vortex causes it to be axially symmetric, and the roll-up process is only a function of radius. The analysis of the isolated wing tip does not concern itself with the time-dependent process that occurs from the trailing edge of the wing to the location where the vortex can be considered rolled up or fully developed. It relates the structure of the vortex sheet at one instant of time (i.e., the trailing edge of the wake-generating wing) to the vortex structure at another time (i.e., the rolled-up vortex far behind the wingtip, which is a steady-state structure).

Wake Model for Wings of Finite Span

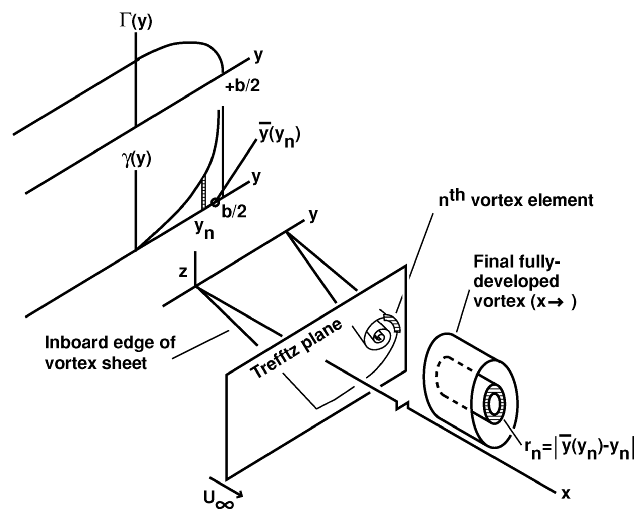
Because a finite wing has its two wing tips in close proximity to each other, the roll up of the two vortices in the pair interacts so that the streamlines and vorticity distributions are no longer axially symmetric (Fig. 1b). From a theoretical point of view, the proximity of the two vortices makes it possible to establish two different streamline patterns that depend on whether the vortices are held in place (Fig. 2a) or allowed to move downward under the influence of their self-induced velocity field (Fig. 2b). The two models are now discussed to illustrate how wake structure can affect vorticity distributions and the complexity of an analysis.

Vortex Pair Held Stationary

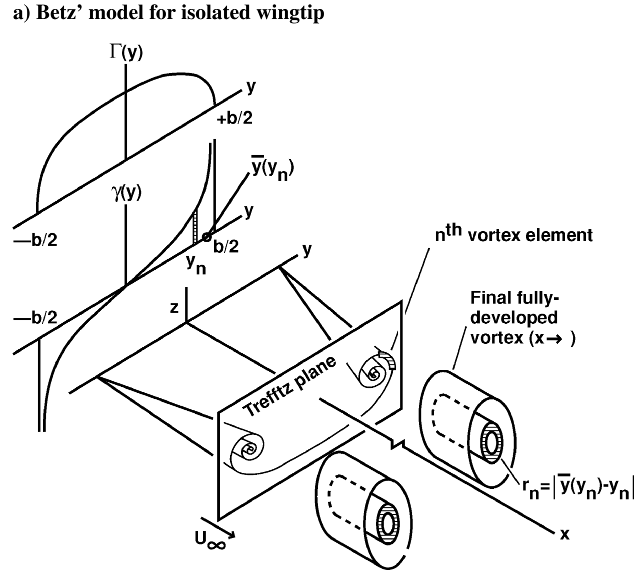
When a pair of point vortices is forcibly held in place, the streamline pattern consists of circular streamlines that have their centers offset by increasing amounts as the radius of the circles becomes larger; Fig. 2a. If the circulation is distributed in the fluid between two particular streamlines, as illustrated by the cross-hatched regions in Fig. 2a, the vorticity or circulation associated with the flowfield moves along a path that carries it from near the centerline to large distances away from it. The circulation is thereby distributed over a large part of the flowfield in a way that is not axially symmetric about each streamline center. Also, the centroid of vorticity is now not located at the center of a circular streamline, but at greater spanwise distances from the wake centerline, where the fluid is moving slowly. As a consequence, when vorticity is distributed throughout the flowfield, the circular streamline pattern, shown for a pair of point vortices in Fig. 2a, will become noncircular. As a result, any analysis or simulation of the flowfield then becomes much more complicated. Fortunately, because the point-vortex pair illustrated in Fig. 2a is not free to move it does not represent the flowfields of vortex wakes of aircraft and need not be discussed further.

Vortex Pair Free to Move

If a vortex pair is not forcibly held stationary, it will move downward under its self-induced velocity field, and the flowfield will be unsteady with time for an observer fixed in space. In a uniform atmosphere the self-induced downward velocity w_{pr} of the vortex pair depends on the total circulation in each vortex and on the spanwise spacing between the centroids of vorticity, which is constant with time in an inviscid fluid. The spanwise distance between the

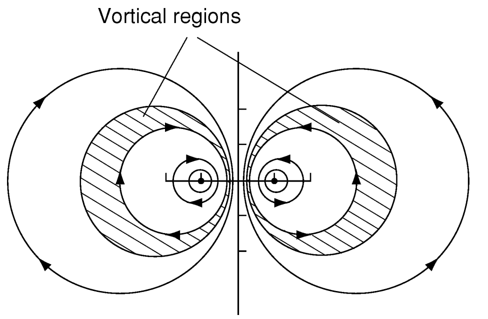


a) Betz' model for isolated wingtip

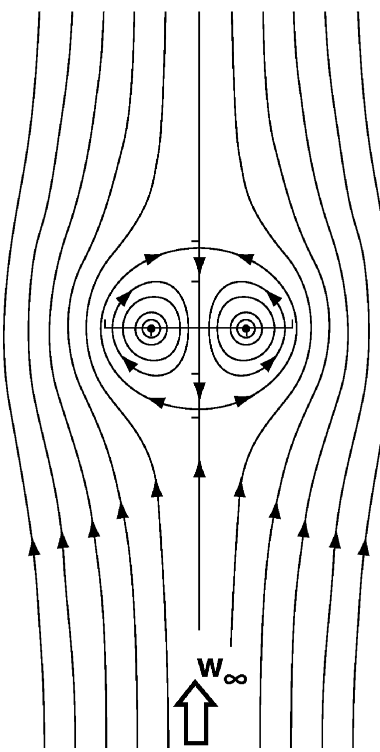


b) Vortex pair shed by finite wingspan

Fig. 1 Diagram for relationship between span loading on wing and final rolled-up vortex structure.



a) Vortices forcibly held in place



b) Vortices moving under self-induced velocity field

Fig. 2 Streamlines for the flowfield of two equal and opposite point vortices: centers designated by filled symbols.

centroids is usually slightly less than the span of the wing that generated the wake. For an elliptically loaded wing the spanwise spacing is given by $b' = \pi b/4$ and the downward velocity by

$$w_{\mathrm{pr}} = -\Gamma_o/2\pi b' \tag{1}$$

where Γ_o is the magnitude of the total circulation in the wake on each side of the centerline, which corresponds to the centerline circulation bound in the wing.

To establish a steady-state flowfield in the reference frame of the coordinate system, as shown in Fig. 2b for a pair of point vortices, an equal and opposite upward velocity $w_\infty = -w_{\mathrm{pr}}$ is superimposed on the flowfield of the two point vortices. The resulting flowfield then consists of a stationary vortex oval embedded in a uniform upwardly moving stream. The oval-shaped boundary now contains all of the circulation shed by the lifting wing that generated the wake. The vortex oval for a pair of point vortices is about 2.09 b' in the spanwise direction and 1.73 b' in the vertical.

Invariants for Motion of Vortices

The analysis to follow is based on the invariants for the time-dependent motion of two-dimensional vortex systems in an incompressible and inviscid fluid.[5,8,9] The invariants are used because they can be set up to relate a given function or property of the vortex system at a given time (e.g., at the wing trailing edge) to the state of the same system at another time (e.g., at a station far behind the wing) without knowledge of the intervening process. Although texts[8,9] often present the invariants in terms of point-vortex systems, the analysis to follow uses the invariants in an integral form to represent more accurately the vorticity distributions in multiple vortex wakes. Because the angular moment of circulation is not used in the analysis, it will not be discussed.[5]

Conservation of Circulation

Although the vortex sheet at the trailing edge of the generating wing is usually not perfectly flat, its vertical location is approximated here by $z_w = 0$, as its original location in the wing plane, to simplify the analysis. Because the strength of the vortex sheet $\gamma_w(y)$ is the circulation content per unit of length of the vortex sheet, it depends on the spanwise gradient of the circulation bound in the wing $\Gamma_w(y)$ as

$$\gamma_w(y) = -\frac{\mathrm{d}\Gamma_w(y)}{\mathrm{d}y} \tag{2a}$$

where the minus sign is necessary for the starboard side of the wing because convention dictates that the strength of the vortex sheet be positive and the gradient of the bound circulation is negative along the starboard side of the wing, decreasing to zero at $y_w = b/2$ (Figs. 1 and 3).

In the circular vortical regions used here to approximate the vorticity distributions in the rolled-up vortical regions of multiple vortex wakes, the circulation and vorticity are related by

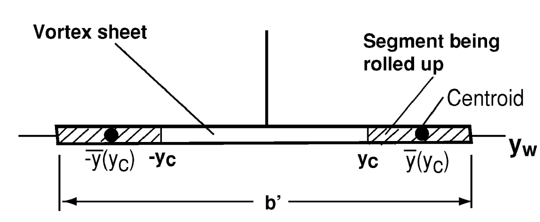
$$\gamma_v(r) = \frac{1}{2\pi r}\frac{\mathrm{d}\Gamma_v(r)}{\mathrm{d}r} \tag{2b}$$

All of the circulation contained in each of the two vortex sheet segments (i.e., port and starboard of the wing centerline) in the plane of the wing is assumed to be contained inside of the two circles of radius r_c in the rolled-up plane of the vortex pair (Fig. 3). That is, it is assumed that the magnitude of the circulation in the vortex sheet passes unchanged into the circulation that enters the rolled-up vortex, that is, circulation is conserved. Therefore,

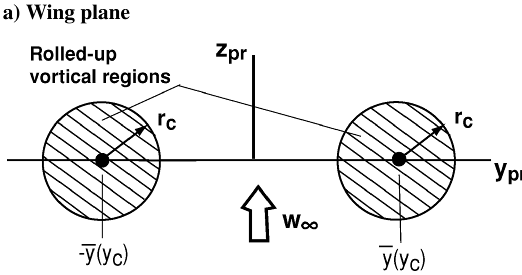
$$\Gamma_w(y_c) = -\int_{y_c}^{b/2}\frac{\mathrm{d}\Gamma_w(y)}{\mathrm{d}y}\,\mathrm{d}y \tag{3a}$$

where $\Gamma_w(b/2) = 0$ and

$$\Gamma_v(r_c) = \int_0^{r_c}\frac{\mathrm{d}\Gamma_v(r)}{\mathrm{d}r}\,\mathrm{d}r \tag{3b}$$



a) Wing plane



b) Plane of vortex pair

Fig. 3 Nomenclature used for roll up of vortex sheets into axially symmetric vortical structures.

where $\Gamma_v(0) = 0$. The subscript w denotes the vortex sheet at the trailing edge of the wing, and the subscript v denotes the downstream station where an isolated vortex sheet is fully rolled up. The parameters y_c and r_c are then corresponding locations on the vortex sheet (or on the span loading) and in the rolled-up vortex where both intervals $\mathrm{d}y_c$ and $\mathrm{d}r_c$ contain the same amount of circulation in their respective flowfields. In the analysis the locations y_c and r_c are used to track the circulation transferred from the vortex sheet to the rolled-up vortex. The radius r_c then defines the radius of the circular vortical region that encloses the circulation between $y = y_c$ and $y = b/2$ transferred from the vortex sheet to the rolled-up vortex (Fig. 3). Because the conservation of circulation specifies that no circulation is lost or gained in the transfer process, Eq. (3) yields

$$\Gamma_w(y_c) = \Gamma_v(r_c) \tag{4}$$

because the bound circulation at the wing tip and at the vortex center are both zero.

First Moment of Circulation

The conservation of the first moment of circulation specifies that the horizontal and vertical locations of the centroids of circulation for the vortex sheet remain fixed. The location of the centroid of circulation is given by

$$\bar{y}(y_c)\Gamma_w(y_c) = \int_{y_c}^{b/2} \frac{y\,\mathrm{d}\Gamma_w(y)}{\mathrm{d}y}\,\mathrm{d}y \tag{5a}$$

$$\bar{z}(z_c)\Gamma_w(z_c) = \int_{y_c}^{b/2} \frac{z\,\mathrm{d}\Gamma_w(y)}{\mathrm{d}y}\,\mathrm{d}y \tag{5b}$$

where it is assumed that the flowfield is in a steady or nonvarying state relative to the coordinate system by a uniform upward velocity. The conservation of the first moment of circulation is satisfied by placing the center, or centroid, of the circulation in the rolled-up plane at the same spanwise location it had in the plane of the vortex sheet.

Second Moment of Circulation

The second moment of circulation is determined about the centroid of circulation in both the wing and the rolled-up vortex planes. As explained in the preceding section, the centroid of circulation has the same spanwise location in both planes. The second moment is again directly proportional to the circulation, but its magnitude is also determined by the summation of the square of the distances between the the elements of circulation (or vorticity) and the centroid of circulation. For a vortex sheet the equation becomes

$$J_w(y_c) = -\int_{y_c}^{b/2} [\bar{y}(y_c) - y]^2 \frac{\mathrm{d}\Gamma_w(y)}{\mathrm{d}y}\,\mathrm{d}y \tag{6a}$$

In Eq. (6a) the distance from each vortex element to the centroid of circulation is specified by $[\bar{y}(y_c) - y]$ in the wing plane (because $z = 0$ on the vortex sheet). The quantity $\bar{y}(y_c)$ is the spanwise location of the centroid of circulation for the vortex sheet segment contained in $y_c \le y_w \ge b/2$. When the circulation is distributed symmetrically about a vortex center, the radial gradient in circulation is used in the calculation of the invariant as

$$J_v(r_c) = \int_0^{r_c} r^2 \frac{\mathrm{d}\Gamma_v(r)}{\mathrm{d}r}\,\mathrm{d}r \tag{6b}$$

If the second moment of circulation is invariant with time, its magnitude in both the plane of the vortex sheet and in the plane of the rolled-up vortex must be the same, that is, $J_v(r_c) = J_w(y_c)$. The second moment of circulation is zero for all antisymmetric vortex systems. For this reason it does not provide any information on the structure of rolled-up vortex pairs. Therefore, the study reported here found it necessary to investigate the possible use of other vortex invariants for the analysis and found that the invariant for energy was the only one that could provide the needed relationships.

Kinetic Energy in Flowfield

The invariant for kinetic energy is more configuration dependent than the foregoing invariants. As a result, various relationships for the energy content of the flowfield will be presented. A method based on contour integration for the kinetic energy in the irrotational part of the flowfield[10] will not be described because it proved not to be useful.

Kinetic Energy by Integration of Velocity

The kinetic energy T induced in the flowfield by vortical regions is found here by integration of the square of the velocity over the entire flowfield as

$$T = \frac{\rho}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v^2 + w^2)\,\mathrm{d}y\,\mathrm{d}z \tag{7}$$

where the density of the fluid and the factor of one-half are needed to complete the energy expression. The velocity components needed for Eq. (7) are given by

$$v(y, z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\gamma(y', z')(z - z')\,\mathrm{d}y'\,\mathrm{d}z'}{(y - y')^2 + (z - z')^2} \tag{8a}$$

$$w(y, z) = +\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\gamma(y', z')(y - y')\,\mathrm{d}y'\,\mathrm{d}z'}{(y - y')^2 + (z - z')^2} \tag{8b}$$

where the integrals again extend over the entire flowfield where the vorticity $\gamma(y', z')$ is nonzero.

Kinetic Energy Induced by Isolated Vortex

The kinetic energy in the flowfield surrounding an isolated point vortex is given by

$$T_{\mathrm{ptvr}} = \frac{\rho}{2}\int_0^{r_{\max}}\int_0^{2\pi} v_\theta^2 r\,\mathrm{d}r\,\mathrm{d}\theta \tag{9a}$$

where the radial integration goes from the center of the vortex throughout the flowfield to $r_{\max} = \infty$. The so-called self-energy for a point vortex is then determined as

$$T_{\mathrm{ptvr}} = \frac{\rho}{2}\Gamma_{\mathrm{ptvr}}^2 \int_0^{r_{\max}} \frac{\mathrm{d}r}{2\pi r} \tag{9b}$$

As the upper limit of integration at $r_{\max}$ is allowed to go to infinite radius, the energy given by the upper limit of integration goes to $+\infty$. At the lower limit of integration, where r goes to zero, the quantity goes to $-\infty$, so that the net energy of an isolated point vortex is equal to $+2\infty$. Because this energy increment (i.e., the self-energy of the point vortex) is constant with time, it is usually not included in the computations.[9,11]

If, however, the isolated vortical region has a circulation content $\Gamma_v(r)$, which is spread over a core of finite radius r_c, the kinetic energy in the flowfield is composed of

$$T_v = \frac{\rho}{2}\int_0^{r_c} \frac{\Gamma_v(r)^2\,\mathrm{d}r}{2\pi r} + \frac{\rho\Gamma_v(r_c)^2}{2}\int_{r_c}^{\infty} \frac{\mathrm{d}r}{2\pi r} \tag{10}$$

where the first radial integration determines the kinetic energy in the rotational core of the vortex and the second integral includes the irrotational part of the flowfield. The combination of the two represents the self-energy of a vortex with a vortical core of finite size.

$$T_v = \frac{\rho}{2}\int_0^{r_c} \frac{\Gamma_v(r)^2\,\mathrm{d}r}{2\pi r} - \frac{\rho}{2}\frac{\Gamma_v(r_c)^2}{2\pi}\ln r_c \tag{11}$$

Because the term at $r = \infty$ is constant, it is usually ignored.[11] Problems with this and other infinite contributions that must be reasoned away are avoided in the present analysis by setting up relationships for gradients that guide the roll-up process and that remain finite over the regions where they are applied.

Kinetic Energy by Kirchhoff–Routh Path Function

The Kirchhoff–Routh path function expresses the energy in a flowfield driven by a system of point vortices and other energy sources. The expression used in the determination of energy is the interactive relationship for the mutually induced energy brought about by the summation of all of the combinations of pairs of point vortices.[8,9]

$$W = -\frac{\rho}{8\pi}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Gamma_i\Gamma_j \ln\left[(y_i - y_j)^2 + (z_i - z_j)^2\right] = \text{constant} \tag{12}$$

where Γ denotes the strength or circulation in each. Note that the summation indices do not include the self-energy of the vortices because they are infinite and are being ignored. A detailed discussion of the energy invariant and how to treat various energy sources is described by Lin.[9] To differentiate between the total energy T, which includes the infinite and constant parts along with the variable parts, the invariant for energy for a point-vortex system is sometimes written only for the variable parts in point-vortex form as W, which is often referred to as the Kirchhoff–Routh path function, after the developers of the function.[8,9] Although both positive and negative signs have been used for W, the negative sign is correct because the infinitely large contributions, which are not included in Eqs. (11) or (12), are positive and much larger than the variable and finite parts of the energy, which are negative.

When the circulation is spread over the flowfield in a continuous manner, the Kirchhoff–Routh path function is determined by the integrals

$$T = \frac{\rho}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma_i\gamma_j \ln\left[(y_i - y_j)^2 + (z_i - z_j)^2\right] \times \mathrm{d}y_i\,\mathrm{d}z_i\,\mathrm{d}y_j\,\mathrm{d}z_j \tag{13}$$

Because the energy determined by Eq. (13) does not require a determination of the velocity components in the fluid and because the energy in the flowfield can be divided into the source combinations generating the energy, it is used in the analysis to follow. Note that the energy contributions are obtained without any knowledge of or specifications on the streamline patterns that are being induced by the vortical regions, that is, the streamlines and the paths of the vortical elements need not be aligned in order to compute the energy in the flowfield.

Kinetic Energy Associated with Freestream Velocity

As mentioned earlier, in order to establish a steady-state flowfield an upwardly moving freestream is added to the flowfield to offset the self-induced downward velocity of the vortex pair. The magnitude of the freestream velocity is made just strong enough to hold the vortex pair stationary, so that the flowfield becomes steady state. The contribution to the kinetic energy by the upwardly moving freestream is calculated by use of a formulation presented by Lin[9] as

$$\Delta T_{fs} = -\rho\sum_{i=1}^{N}\gamma_i\Psi_c(y_i, z_i) \tag{14}$$

where

$$\Psi_c(y_i, z_i) = \sum_{i=1}^{N} y_i w_\infty \tag{15}$$

The form of Eq. (14) is such that when a derivative is taken of $\Psi_c(y_i, z_i)$ the velocity for the vertical freestream results. The function can then be put in a simpler form by observing that w_∞ can be written as

$$w_\infty = -\frac{\Gamma_w(y_c)}{4\pi\bar{y}}$$

so that the kinetic energy contributed by the freestream is given by

$$T_{fs} = \rho\frac{\Gamma_w(y_c)^2}{4\pi} \tag{16}$$

where $\Gamma_w(y_c)$ is the total circulation transferred to the rolled-up vortex thus far in the process. The same quantity also applies to the energy of the vortices in the rolled-up plane.

Roll Up of Isolated Vortex Sheet

Two derivations of the roll-up equation for an isolated vortex (Fig. 1) are now presented as practice cases to more thoroughly identify the best procedures to be used for the derivation of the roll-up equations for vortex pairs. The first case treats the derivation of the roll-up equations by use of the invariants for conservation of circulation and the first and second moments of circulation. The second case is similar to the first except that the invariant for energy is used in the derivation instead of the invariant for the second moment of circulation. In both cases the roll-up sequence and the axially symmetric structure of the rolled-up vortex are as assumed in Betz' derivation.[1,2] Because the two results should then be in exact agreement, the proper technique to be used to derive roll-up relationships for vortex pairs is developed. As indicated in the text to follow, it is found that it is the rate of change of the invariants with the size of the vortical region, and not the value for the entire vortical region, that provides their needed relationship for the structure of the rolled-up vortices.

By Invariant for Second Moment

Because the circulation in the vortex sheet is the same as the circulation in the rolled-up vortex at each step of the transfer, the conservation of circulation yields

$$\Gamma_w(y_c) = \Gamma_v(r_c) \tag{17}$$

where y_c and r_c are corresponding points on the vortex sheet and in the rolled-up vortex. The conservation of the first moment of circulation specifies that the location of the centroid of circulation in the plane of the rolled-up vortex must be the same as it was in the plane of the vortex sheet at each stage of the transfer of vortex seqments. That is, the center of the vortex (i.e., its centroid) must be located so that $r = 0$ at $y = \bar{y}(y_c)$. The second moment of circulation must then also be taken about the same centroid location, so that by Eqs. (6)

$$J_w(y_c) = -\int_{y_c}^{b/2}[\bar{y}(y_c) - y]^2\frac{\mathrm{d}\Gamma_w(y)}{\mathrm{d}y}\,\mathrm{d}y \tag{18a}$$

$$J_v(r_c) = \int_0^{r_c} r^2\frac{\mathrm{d}\Gamma_v(r)}{\mathrm{d}r}\,\mathrm{d}r \tag{18b}$$

A relationship between the inboard end of the vortex sheet y_c and the outer radius of the rolled-up vortical region is found by calculating the rate at which circulation is transferred from one plane to another. The two rates are calculated as

$$\frac{\mathrm{d}J_w(y_c)}{\mathrm{d}y_c} = [\bar{y}(y_c) - y_c]^2\frac{\mathrm{d}\Gamma_w(y_c)}{\mathrm{d}y_c} - 2\frac{\mathrm{d}\bar{y}(y_c)}{\mathrm{d}y_c} \times\int_{y_c}^{b/2}[\bar{y}(y_c) - y]\frac{\mathrm{d}\Gamma_w(y)}{\mathrm{d}y}\,\mathrm{d}y$$

Because $\mathrm{d}\bar{y}(y_c)/\mathrm{d}y_c$ is constant, it is placed outside of the integral. The integral is then zero because both terms yield $\bar{y}(y_c)\Gamma_w(y_c)$, but because they are of opposite sign the integral vanishes. The differentials of the second moment of circulation in the wing plane, and in the rolled-up plane, then become

$$\frac{\mathrm{d}J_w(y_c)}{\mathrm{d}y_c} = [\bar{y}(y_c) - y_c]^2\frac{\mathrm{d}\Gamma_w(y_c)}{\mathrm{d}y_c} \tag{19a}$$

$$\frac{\mathrm{d}J_v(r_c)}{\mathrm{d}r_c} = r_c^2\frac{\mathrm{d}\Gamma_v(r)}{\mathrm{d}r_c} \tag{19b}$$

Conservation of circulation specifies that the rate at which circulation is transferred from the wing plane to the rolled-up plane be the same. From Eq. (17)

$$\frac{\mathrm{d}\Gamma_w(y_c)}{\mathrm{d}y_c}\,\mathrm{d}y_c = \frac{\mathrm{d}\Gamma_v(r_c)}{\mathrm{d}r_c}\,\mathrm{d}r_c \tag{20}$$

Combination of Eqs. (18–20) results in the Betz relationship.[1,2]

$$r_c = \bar{y}(y_c) - y_c \tag{21}$$

By Invariant for Energy

The same procedure used for the derivation of Betz' relationship in the preceding section is now applied to the same set of equations, except the invariant for energy is used instead of the second moment of circulation. From Eq. (13) the kinetic energy content in the flowfield of the vortex sheet shown for the starboard side of the wake in Fig. 3 can be written for the vortex sheet as

$$T_w = -\frac{\rho}{8\pi}\int_{y_c}^{b/2}\int_{y_c}^{b/2}\frac{\mathrm{d}\Gamma_i}{\mathrm{d}y_i}\frac{\mathrm{d}\Gamma_j}{\mathrm{d}y_j}\,\ln\left[(y_i - y_j)^2\right]\mathrm{d}y_i\,\mathrm{d}y_j \tag{22a}$$

Similarly, the kinetic energy in the flowfield of the rolled-up vortex is given by

$$T_v = \frac{\rho}{2}\left[\int_0^{r_c}\frac{\Gamma_v(r)^2\,\mathrm{d}r}{2\pi r} - \frac{\Gamma_v(r_c)^2}{2\pi}\ln r_c\right] \tag{22b}$$

Because the variable parts of the energy invariants in the two flowfields must be equal, their differentials must also be equal. The differentials are found as

$$\frac{\mathrm{d}T_w}{\mathrm{d}y_c} = \frac{\rho}{2\pi}\frac{\mathrm{d}\Gamma_c}{\mathrm{d}y_c}\int_{y_c}^{b/2}\frac{\mathrm{d}\Gamma_i}{\mathrm{d}y_i}\ln[(y_i - y_c)]\,\mathrm{d}y_i \tag{23a}$$

where y_i is always greater than y_c. Similarly, the differential for the energy in the flowfield of the rolled-up vortex is given by

$$\frac{\mathrm{d}T_v}{\mathrm{d}r_c} = -\frac{\rho}{2\pi}\left[\Gamma_v(r_c)\frac{\mathrm{d}\Gamma_v(r_c)}{\mathrm{d}r_c}\,\mathrm{d}r_c\,\ln r_c\right] \tag{23b}$$

because two of the terms cancel each other. The two functions given by Eqs. (23a) and (23b) appear at first not to be equal. However, the streamfunction for the streamline that passes through the inboard end of the vortex segment y_c in the plane of the vortex sheet is noted to be given by

$$\Psi_w(y_c) = \frac{\rho}{2\pi}\int_{y_c}^{b/2}\frac{\mathrm{d}\Gamma_w(y_i)}{\mathrm{d}y_i}\ln[(y_i - y_c)]\,\mathrm{d}y_i \tag{24a}$$

which is also equal to

$$\Psi_w(y_c) = \frac{\rho\Gamma_w(y_c)}{2\pi}\ln[\bar{y}(y_c) - y_c] \tag{24b}$$

Eq. (23a) can then be written as

$$\frac{\mathrm{d}T_w}{\mathrm{d}y_c} = \frac{\rho}{2\pi}\Gamma_w(y_c)\frac{\mathrm{d}\Gamma_w(y_c)}{\mathrm{d}y_c}\ln[(\bar{y}(y_c) - y_c)] \tag{25}$$

Comparison of Eq. (23b) with Eq. (25) again leads to the Betz relationship as given by Eq. (21).

This result confirms that the invariant for kinetic energy yields the same structure for rolled-up vortices, as does the equation derived by use of the invariant for the second moment of circulation, that is, the Betz relationship. Therefore, the computations carried out previously,[5] which appeared to be in approximate agreement, should have been in exact agreement. The differences between the two results are now attributed to computational inaccuracies induced by the difference formula based on point vortices, which was used in that paper to compute those results.

Roll Up of Vortex Pairs

The procedures used in the forgoing derivations are now applied to the more difficult flowfield of a vortex pair. The nomenclature illustrated in Fig. 3 is again used here. To simplify the computation of kinetic energy, the vortical regions are again assumed to be axially symmetric about the centroids, or centers, of the distributions, even though they are not closely aligned everywhere with the streamline pattern (Fig. 4). Both centroids in the plane of the vortex pair are at the same spanwise locations as occupied by the two centroids in the wing plane.

The Kirchhoff–Routh formulation for the kinetic energy in the flowfield allows the energy to be divided into the self-energy of each vortical region and the interactive energy of the two vortical regions. Because the expression for energy only contains elements of two vortical regions, it is not necessary to treat combinations that involve more than two vortical regions. The kinetic energy in the plane of both the vortex sheet and the rolled-up vortex cores can both be divided into three parts: namely, the self-energy of the port and of the starboard vortical regions and the interactive energy of the two. From solutions presented in preceding sections, the solutions are then already known for the self-energy of each of the two vortical regions in the planes of the vortex sheets and for their rolled-up counterparts because they can all be treated as isolated vortical regions. The port and starboard vortical regions in both planes have the same energy content because the self-energy is independent of the sign of vorticity. It only remains, therefore, to determine the interactive energy for two vortex sheets in the wing plane and the interactive energy for the two circular vortical regions in the plane of the vortex pair.

Kinetic Energy Induced by Interaction of Two Vortex Sheets

The integral for the energy of interaction between two vortex sheets differs from the one used for the computation of self-energy induced by the starboard vortex sheet in that the vortical regions are of opposite sign and the distance between the vortex elements being analyzed lie on opposite sides of the wake centerline. The distance between vortex elements then becomes $(y_i + y_j)$ rather than $(y_i - y_j)$. Also, because the vortex strengths in the port and starboard vortex sheets are opposite in sign, a sign change is also required. The kinetic energy contribution to the flowfield brought about by the interaction of the two vortex sheets can then be written as

$$T_w = +\frac{\rho}{8\pi}\int_{y_c}^{b/2}\int_{y_c}^{b/2}\frac{\mathrm{d}\Gamma_i}{\mathrm{d}y_i}\frac{\mathrm{d}\Gamma_j}{\mathrm{d}y_j}\ln[(y_i + y_j)^2]\,\mathrm{d}y_i\,\mathrm{d}y_j \tag{26}$$

When the rate of change in the energy contributed by the across-centerline interaction of the two vortical regions is evaluated as a

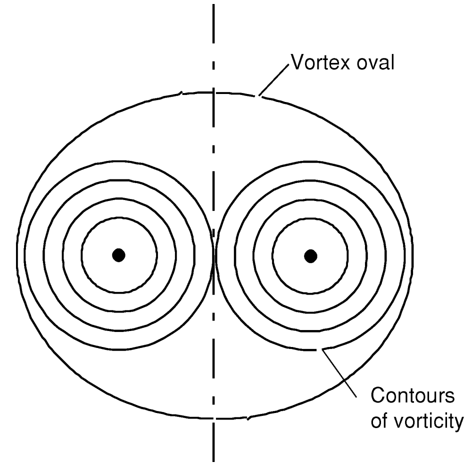


Fig. 4 Model used for arrangement of vortical regions inside vortex oval to facilitate computation of kinetic energy in flowfield of vortex pair.

function of the length of the sheet segment given by y_c, Eq. (26) becomes

$$\frac{\mathrm{d}T_w}{\mathrm{d}y_c} = -\frac{\rho}{2\pi}\Gamma_w(y_c)\,\frac{\mathrm{d}\Gamma_w(y_c)}{\mathrm{d}y_c}\,\ell\mathrm{n}[2\bar{y}(y_c)] \tag{27}$$

Note that the interactive energy does not influence the distribution of vorticity within the vortical regions, but only specifies that the centroids of the two vortical regions must be separated by the same distance in the two planes. Such a specification is the same as the one placed on the vortex pair by the first moment of circulation. As a consequence, not only is it possible to divide the energy content in the flowfield of the vortex sheets into self-energy and interactive energy, but the two parts each govern a different aspect of the rolled-up flowfield.

Kinetic Energy Induced by Interaction of Two Axially Symmetric Vortical Regions

The kinetic energy contributed by the interaction of two circular vortical regions is obtained by integration of the square of the velocity throughout the flowfield of the vortex pair by use of the integral

$$T_{\mathrm{pr}} = -\frac{\rho}{8\pi}\int_0^{r_c}\int_0^{r_c}\int_0^{2\pi}\int_0^{2\pi}\gamma_i(r_i,\theta_i)\gamma_j(r_j,\theta_j)$$
$$\times\,\ell\mathrm{n}\Big[r_i^2 + r_j^2 + 4\bar{y}^2 + 4\bar{y}(r_i\cos\theta_i - r_j\cos\theta_j)$$
$$+\,2r_ir_j\cos(\theta_i-\theta_j)r_i\,\mathrm{d}\theta_i r_j\,\mathrm{d}\theta_j\,\mathrm{d}r_i\,\mathrm{d}r_j\Big] \tag{28}$$

where the $\gamma_i(r_i,\theta_i)$ quantities represent the circulation content of the various integration areas in the flowfield. Integration of Eq. (28) over the entire flowfield of the two axially symmetric vortical regions, like those shown in Fig. 3, yields

$$T_{\mathrm{pr}} = \frac{\rho}{4\pi}\Gamma(r_c)^2\,\ell\mathrm{n}[2\bar{y}(y_c)] \tag{29}$$

The differential of T_{pr} yields the rate of change of the kinetic energy in the flowfield as a function of the size of the circular vortical regions as

$$\frac{\mathrm{d}T_{\mathrm{pr}}}{\mathrm{d}r_c} = \frac{\rho}{2\pi}\Gamma(r_c)\,\frac{\mathrm{d}\Gamma(r_c)}{\mathrm{d}r_c}\,\ell\mathrm{n}[2\bar{y}(y_c)] \tag{30}$$

which has the same value as obtained for a pair of vortex sheets in Eq. (27).

The foregoing results indicate that when the kinetic energy in the flowfield is divided into self- and interactive-energy components, not only do the energy contributions separate but the role exercised by the energy contributions are also completely separate. That is, the self-energy of each vortical region determines the rolled-up structure, and the interactive energy specifies the distance between the centroids of the two regions. The roles do not overlap, according to the first-order approximation found here. These simple guidelines appear to apply whether the vortex system is composed of only one or a number of vortical regions. Therefore, the analysis also justifies the superposition of axially symmetric vortical regions to simulate multiple vortex wakes and explains why Betz' method does such a good job of representing complex vortical flowfields.

Application to Elliptically Loaded Wing

Before discussing higher-order approximations to the structure of the vortical regions in the rolled-up plane of the vortex pair, an example is presented to make two comparisons that evaluate the accuracy of first-order simulations. As pointed out earlier, the first-order solutions are based on the assumption that the contours of vorticity (or circulation) can be approximated by axially symmetric distributions, even though the streamline pattern might differ from contours of vorticity shown in Fig. 4. The variation of the circulation in the vortical cores shown there, as found by the roll-up guidelines for an elliptically loaded wing, indicate that most of the vorticity is concentrated near the centers of the vortical regions (Fig. 5). For example, about 60% of the circulation is contained within the inner 20% of the core and 80% is contained within the inner 40% of the core.

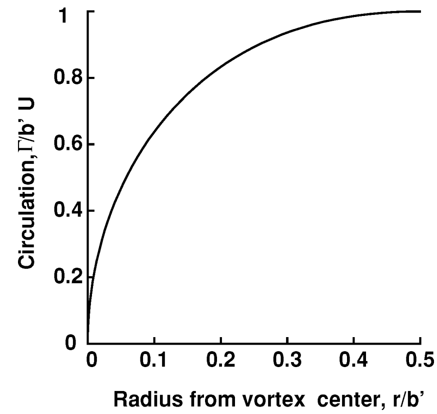


Fig. 5 Variation of circulation in rolled-up vortical region as a function of radius for elliptically loaded wing.

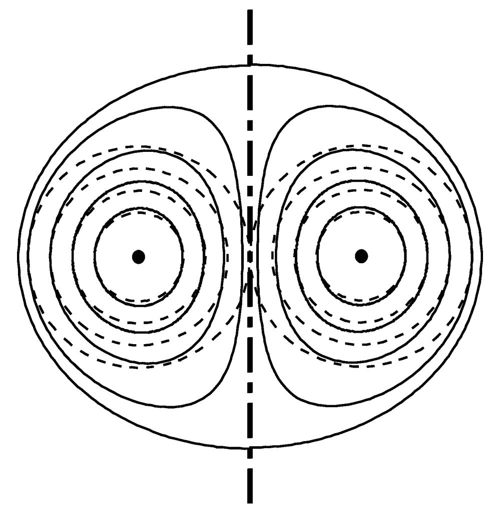


Fig. 6 Comparison of streamline paths (——) with contours of circulation (– – –) used in first-order roll-up method for elliptic spanwise loading on wake-generating wing.

The streamlines for a pair of two such vortical regions are presented in Fig. 6. In the steady-state situation the streamline paths (solid lines) would be the same as the contours of constant circulation (dashed lines). Near the centers of the vortical regions, the two quantities appear to be well aligned, and significant differences are mainly at the tops and bottoms of the contours. It is not too surprising then that the streamlines induced by a pair of point vortices are nearly the same as those induced by the vortical distributions derived by the first-order method for a wing with elliptic span loading (Fig. 7). If the computations had been carried out for a second approximation, the streamlines and contours of circulation would probably have been closely aligned. It is estimated therefore that the first-order approximation is adequate for most engineering purposes. Such a result is not surprising because so much of the circulation is concentrated near the centers as mentioned in the preceding paragraph and shown in Fig. 5.

Higher-Order Approximations

As a reminder, circular or axially-symmetric vortical regions were chosen because the kinetic energy in the flowfield associated with each area is then relatively easy to compute, and the relationships derived are simple. The procedures required for higher-order approximations would first need to adjust the vorticity contours to become aligned with the streamlines obtained by use of the first-order

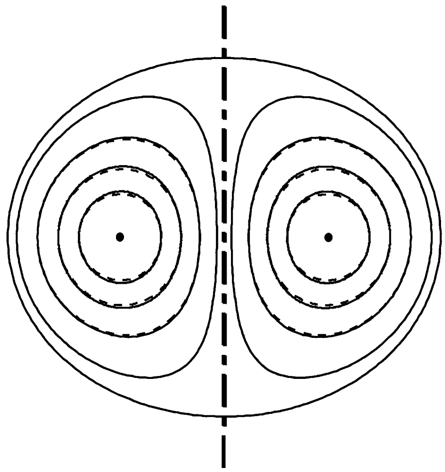


Fig. 7 Comparison of streamline paths for a pair of point vortices (——) with first-order estimate of paths induced by rolled-up vortex pair shed by elliptic spanwise loading on wake-generating wing (– – –). Both methods predict the same shape for the oval and for the streamlines outside of the oval.

approximation. Based on the updated structure of the vortical regions, new streamline paths would be computed. Such a computation will undoubtedly have to be carried out by numerical means because the streamline shapes and energy evaluations would become very complex to represent. Because the comparisons presented in Fig. 7 are quite good, it is suggested that some other approach (which will no doubt be numerical) be used to compute the structure of rolled-up vortex wakes, when a first-order solution is determined to be inadequate.

Application to Wake Measurements

Because direct and complete experimental confirmation of the theoretical results found in the foregoing analysis is not available, the reader is referred to references cited before[3–7] and also to several other articles that tend to confirm the findings presented here.[12,13] First, as mentioned in the Introduction, both the direct-Betz and indirect-Betz methods have already been applied to the analysis of a number of lift-generated wakes with good success.[2–7] In the studies reported Betz' original method and its extensions provide not only reliable predictions for the structure of vortex wakes, but also guidance in the interpretation of measured wake structures. In these applications, which occurred before the research described here was carried out, it was assumed that circular distributions of vorticity (or circulation) provide good theoretical approximations for vortex wakes that contain two or more vortex structures. The analysis presented here justifies such an assumption because it clarifies how the energy in the flowfield associated with each vortical region can be separated into a part that relates only to the distance between each vortical region and a part that relates only to the structure of the circulation within each vortical region. In the references cited, both the structure of each vortical region and the separation distance between vortex centroids have been experimentally confirmed for some time for flowfields that contain one vortex pair.[3–6] A similar confirmation for wakes composed of multiple vortex pairs has only been carried out in an approximate fashion.[7] That is, only the circulation content in the various vortical regions was measured and compared with theoretically estimated values with reasonably good agreement. Although the distances between vortex centers were judged to be about correct for the multiple vortex wakes considered, measurements were not made of the spacings.

One purpose of the foregoing analysis was to determine whether the streamline and vorticity relocations that occur when vortical regions are superimposed have a significant affect on measured swirl velocity distributions in vortices. Of interest are some differences observed in the inboard and outboard velocity profiles measured in wind tunnel behind models of subsonic transport aircraft.[12,13] It is concluded that the differences found are not explained by any interactions between vortical regions because Fig. 7 indicate that the magnitude of any asymmetries would be very small compared with those observed in the experimental results. The lack of symmetry measured in the vortex wakes of aircraft models must then have been caused by some other characteristic of the wake-generating model.

Conclusions

An analysis based on the two-dimensional time-dependent invariants for vortex systems is used to extend Betz' method for isolated vortices to the vortex pairs that are associated with lift-generating wings of finite span. In the analysis it was found that the vortical regions (vortices) in the flowfield, which drive the kinetic energy, can be divided into self- and interactive-energy sources. Not only was the analysis simplified by the observation, but the separation disclosed that the self-energy of each vortical region governs the radial distribution of circulation in its rolled-up vortical core. Similarly, the sources of interactive energy govern the distances between the centroids of vortical cores. Comparison of the streamline patterns for various vortical distributions indicates that first-order estimates based on circular vortical distributions provide surprisingly accurate results that should suffice for many engineering purposes. Therefore, cumbersome higher-order approximations will usually not need to be found. The results of this study also verify the validity of a frequently used procedure that superimposes axially symmetric vortex structures to produce a flowfield composed of multiple vortex pairs.

The theoretical tools derived here apply best to lift-generated flowfields within several wing spans behind the wake-generating aircraft, before wake instabilities and viscous and turbulent diffusion have taken place. As time and distance behind the wake-generating aircraft increase, the wake structure will deviate more and more from those predicted by the simple, inviscid theory presented here.

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